Analysis of Q-Learning: Switching System Approach

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Abstract

Q-learning is known to be one of the most popular reinforcement learning algorithms to find an optimal policy for an unknown Markov decision process. In this paper, we introduce a new asymptotic convergence analysis of Q-learning based on switching system perspectives and theories. The approach provides a unified viewpoint and greatly simplifies the analysis for a large family of Q-learning algorithms.

1 Introduction

Q-learning, originally introduced by Watkin in [15], is one of the most popular and fundamental reinforcement learning (RL) algorithms for finding the optimal policy of unknown Markov decision processes. There exist few approaches that prove the asymptotic convergence of Q-learning: the original proof [15],the stochastic approximation and contraction mapping-based approach [5, 13], and the stochastic approximation and ODE (ordinary differential equation) approach [2].

The ODE approach analyzes the convergence of general stochastic recursions by examining stability of the associated ODE model [1, 7, 2] and has been used as a convenient analysis tool to prove convergence of many RL algorithms. However, its application to Q-learning has been limited due to the presence of the max operator, which makes the associated ODE model a complex nonlinear system. In contrast, the associated ODE of TD-learning [12] for policy evaluation is linear, whose asymptotic stability is easier to analyze in general. While [2] gave the convergence proof of Q-learning based on a nonlinear ODE model, to the authors' knowledge, substantial analysis is required to prove the stability of the corresponding nonlinear ODE [3] by using the max-norm contraction of the Bellman operator. Moreover, the stability analysis does not immediately extend to other Q-learning variants, Q-learning with linear function approximation, distributed Q-learning, and averaging Q-learning [8].

In this paper, we study a simple and unified framework to analyze Q-learning through switching linear system (SLS) models [9] of the associated ODE. SLSs are an important class of nonlinear hybrid systems, where the system dynamics matrix varies within a finite set of subsystem matrices (or modes) according to a switching signal. The study of SLSs has attracted much attention in the past (see [10] and [9] for comprehensive study and surveys). We show that a nonlinear ODE model associated with Q-learning can be formulated as an SLS, and analyze its asymptotic stability by leveraging particular structure of Q-learning, switching system theories [10, 9], and nonlinear control theories [6]. This switching system approach can be easily extended to other Q-learning variants, such as Q-learning with linear function approximation, distributed Q-learning, and averaging Q-learning [8]. Due to page limits, we only focus on the analysis of the standard Q-learning algorithm here.

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2 Preliminaries

2.1 Markov decision problem

In this paper, we consider the infinite-horizon (discounted) Markov decision problem (MDP), where the agent sequentially takes actions to maximize cumulative discounted rewards. In a Markov decision process with the state-space $\mathcal{S} := \{1, 2, \ldots, |\mathcal{S}|\}$ and action-space $\mathcal{A} := \{1, 2, \ldots, |\mathcal{A}|\}$, the decision maker selects an action $a \in \mathcal{A}$ with the current state s, then the state transits to s' with probability $P_a(s, s')$, and the transition incurs a random reward $r_a(s, s')$, where $P_a \in \mathbb{R}^{|\mathcal{S}|\times|\mathcal{S}|}, a \in \mathcal{A}$, $P_a(s, s')$ is the state transition probability from the current state $s \in \mathcal{S}$ to the next state $s' \in \mathcal{S}$ under action $a \in \mathcal{A}$, and $r_a(s, s')$ is the reward random variable conditioned on $a \in \mathcal{A}, s, s' \in \mathcal{S}$ with its expectation $\mathbb{E}[r_a(s, s')|s, a, s'] = R_a(s, s')$. A deterministic policy, $\pi : \mathcal{S} \to \mathcal{A}$, maps a state $s \in \mathcal{S}$ to an action $\pi(s) \in \mathcal{A}$. The Markov decision problem (MDP) is to find a deterministic optimal policy, π^*, such that the cumulative discounted rewards over infinite time horizons is maximized, i.e.,

$$\pi^* := \operatorname*{arg\,max}_{\pi\in\Theta} \mathbb{E}\left[\left.\sum_{k=0}^{\infty} \alpha^k r_{a_k}(s_k, s_{k+1})\right| \pi\right],$$

where $\gamma \in [0, 1)$ is the discount factor, Θ is the set of all admissible deterministic policies, $(s_0, a_0, s_1, a_1, \ldots)$ is a state-action trajectory generated by the Markov chain under policy π, and $\mathbb{E}[\cdot|\cdot, \pi]$ is an expectation conditioned on the policy π. The Q-function under policy π is defined as

$$Q^\pi(s, a) = \mathbb{E}\left[\left.\sum_{k=0}^{\infty} \gamma^k r_{a_k}(s_k, s_{k+1})\right| s_0 = s, a_0 = a, \pi\right], \quad s \in \mathcal{S}, a \in \mathcal{A},$$

and the corresponding optimal Q-function is defined as $Q^*(s, a) = Q^{\pi^*}(s, a)$ for all $s \in \mathcal{S}, a \in \mathcal{A}$. Once Q^* is known, then an optimal policy can be retrieved by $\pi^*(s) = \arg\max_{a\in\mathcal{A}} Q^*(s, a)$.

2.2 Basics of nonlinear system theory

Consider the nonlinear system

$$\frac{d}{dt}x_t = f(x_t), \quad x_0 = z, \quad t \geq 0, \tag{1}$$

where $x_t \in \mathbb{R}^n$ is the state and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear mapping. The solution to (1) exists and is unique so long as the mapping f is globally Lipschitz continuous.

Lemma 1 ([6, Theorem 3.2]). *Consider the nonlinear system* (1) *and assume that f is globally Lipschitz continuous, i.e.,*

$$\|f(x) - f(y)\| \leq L\|x - y\|, \quad \forall x, y \in \mathbb{R}^n, \tag{2}$$

for some $L > 0$ and norm $\|\cdot\|$, then it has a unique solution $x(t)$ for all $t \geq 0$ and $x(0) \in \mathbb{R}^n$.

An important concept in dealing with the nonlinear system is the equilibrium point. A point $x = x^e$ in the state space is said to be an equilibrium point of (1) if it has the property that whenever the state of the system starts at x^e, it will remain at x^e [6]. For (1), the equilibrium points are the real roots of the equation $f(x) = 0$. The equilibrium point x^e is said to be globally asymptotically stable if for any initial state $x_0 \in \mathbb{R}^n$, $x_t \to x^e$ as $t \to \infty$.

2.3 Switching system theory

Consider the particular nonlinear system, called the *linear switching system*,

$$\frac{d}{dt}x_t = A_{\sigma_t}x_t, \quad x_0 = z \in \mathbb{R}^n, \quad t \in \mathbb{R}_+, \tag{3}$$

where $x_t \in \mathbb{R}^n$ is the state, $\sigma \in \mathcal{M} := \{1, 2, \ldots, M\}$ is called the mode, and $\sigma_t \in \mathcal{M}$ is called the switching signal, and $\{A_\sigma, \sigma \in \mathcal{M}\}$ are called the subsystem matrices. The switching signal can be either arbitrary or controlled by the user under a certain switching policy. Especially, a state-feedback switching policy is denoted by $\sigma(x_t)$. Now, we provide a vector comparison principle [14, 4, 11] for multi-dimensional O.D.E., which will play a central role in the analysis below. We first introduce the quasi-monotone increasing function, which is a necessary prerequisite for the comparison principle.

Definition 1 (Quasi-monotone function). *A vector-valued function $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f := [f_1 \quad f_2 \quad \cdots \quad f_n]^T$ is said to be quasi-monotone increasing if $f_i(x) \le f_i(y)$ holds for all $i \in \{1, 2, \dots, n\}$ and $x, y \in \mathbb{R}^n$ such that $x_i = y_i$ and $x_j \le y_j$ for all $j \ne i$.*

Lemma 2 (Vector comparison principle [14, page 112], [4, Theorem 3.2]). *Suppose that $\overline{f}$ and $\underline{f}$ are globally Lipschitz continuous. Let v_t be a solution of the system*

$$\frac{d}{dt}x_t = \overline{f}(x_t), \quad x_0 \in \mathbb{R}^n, \forall \quad t \ge 0,$$

assume that $\overline{f}$ is quasi-monotone increasing, and let v_t be a solution of the system

$$\frac{d}{dt}v_t = \underline{f}(v_t), \quad v_0 < x_0, \quad \forall t \ge 0,$$

where $\underline{f}(v) \le \overline{f}(v)$ holds for any $v \in \mathbb{R}^n$. Then, $v_t \le x_t$ for all $t \ge 0$.

For completeness, we provide its proof in Appendix for cases tailored for our interest. Lastly, to prove the global asymptotic stability of the switching system, we will use a fundamental algebraic stability condition of switching systems reported in [10].

Lemma 3 (Global asymptotic stability [10, Theorem 8]). *The origin of the linear switching system (3) is the unique globally asymptotically stable equilibrium point under arbitrary switchings, σ_t, if and only if there exist a full column rank matrix , $L \in \mathbb{R}^{m\times n}$, $m \ge n$, and a family of matrices, $\bar{A}_\sigma \in \mathbb{R}^{m\times n}, \sigma \in \mathcal{M}$, with the so-called 'strictly negative row dominating diagonal condition,' i.e., for each $\bar{A}_\sigma, \sigma \in \mathcal{M}$, its elements satisfying*

$$[\bar{A}_\sigma]_{ii} + \sum\nolimits_{j\in\{1,2,\dots,n\}\setminus\{i\}} |[\bar{A}_\sigma]_{ij}| < 0, \quad \forall i \in \{1, 2, \dots, m\},$$

where $[\cdot]_{ij}$ is the (i, j)-element of a matrix $(\cdot)$, such that the following matrix relations are satisfied:

$$L A_\sigma = \bar{A}_\sigma L, \quad \forall \sigma \in \mathcal{M}.$$

More comprehensive surveys and study of stability of switching systems can be found in [10] and [9].

2.4 ODE-based stochastic approximation

Due to its simplicity, the convergence analysis of many RL algorithms rely on the ODE (ordinary differential equation) approach [1, 7]. It analyzes convergence of general stochastic recursions by examining stability of the associated ODE model based on the fact that the stochastic recursions with diminishing step-sizes approximate the corresponding ODEs in the limit. One of the most popular approach is based on the Borkar and Meyn theorem [2]. We now briefly introduce the Borkar and Meyn's ODE approach [2] for analyzing convergence of the general stochastic recursions

$$\theta_{k+1} = \theta_k + \alpha_k(f(\theta_k) + \varepsilon_{k+1}) \tag{4}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear mapping. Basic technical assumptions are given below.

Assumption 1.

1. *The mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz continuous and there exists a function $f_\infty : \mathbb{R}^n \to \mathbb{R}^n$ such that*
$$\lim_{c\to\infty} \frac{f(cx)}{c} = f_\infty(x), \quad \forall x \in \mathbb{R}^n.$$
2. *The origin in $\mathbb{R}^n$ is an asymptotically stable equilibrium for the ODE $\dot{x}_t = f_\infty(x_t)$.*
3. *There exists a unique globally asymptotically stable equilibrium $\theta^e \in \mathbb{R}^n$ for the ODE $\dot{x}_t = f(x_t)$, i.e., $x_t \to \theta^e$ as $t \to \infty$.*
4. *The sequence $\{\varepsilon_k, \mathcal{G}_k, k \ge 1\}$ with $\mathcal{G}_k = \sigma(\theta_i, \varepsilon_i, i \le k)$ is a Martingale difference sequence. In addition, there exists a constant $C_0 < \infty$ such that for any initial $\theta_0 \in \mathbb{R}^n$, we have $\mathbb{E}[\|\varepsilon_{k+1}\|^2|\mathcal{G}_k] \le C_0(1 + \|\theta_k\|^2), \forall k \ge 0$.*
5. *The step-sizes satisfy $\alpha_k > 0, \sum_{k=0}^\infty \alpha_k = \infty, \sum_{k=0}^\infty \alpha_k^2 < \infty$.*

Lemma 4 (Borkar and Meyn theorem [2]). *Suppose that Assumption 1 holds. For any initial $\theta_0 \in \mathbb{R}^n$, $\sup_{k\ge0} \|\theta_k\| < \infty$ with probability one. In addition, $\theta_k \to \theta^e$ as $k \to \infty$ with probability one.*

3 Revisit Q-learning

In this section, we briefly review the standard Q-learning [15] and introduce an additional assumption adopted in this paper.

The standard Q-learning [15] updates

$$Q_{k+1}(s_k, a_k) = Q_k(s_k, a_k) + \alpha_k(s_k, a_k)\left\{r_{a_k}(s_k, s_{k+1}) + \gamma \max_{a\in\mathcal{A}} Q_k(s_{k+1}, a_k) - Q_k(s_k, a_k)\right\},$$

where $0 \le \alpha_k(s,a) \le 1$ is called the learning rate associated with the state-action pair (s,a) at iteration k. This value is assumed to be zero if $(s,a) \neq (s_k, a_k)$. If $\sum_{k=0}^{\infty} \alpha_k(s,a) = \infty$, $\sum_{k=0}^{\infty} \alpha_k^2(s,a) < \infty$, and every state-action pair is visited infinitely often, then the iterate is guaranteed to converge to Q^* with probability one. Note that the state-action can be visited arbitrarily, which is more general than stochastic visiting rules.

To analyze the convergence based on the switching system model, we consider the stronger assumption that $\{(s_k, a_k)\}_{k=0}^{\infty}$ is a sequence of i.i.d. random variables with a fixed underlying probability distribution, $d_a(s), s \in \mathcal{S}, a \in \mathcal{A}$, of the state and action pair (s,a). This assumption is common in the ODE approaches for Q-learning and TD-learning [12]. Moreover, this assumption can be relaxed by considering a time-varying distribution. However, this direction is not addressed in this paper to simplify the presentation of the proofs.

Throughout the paper, we assume that

Assumption 2. *$d_a(s) > 0$ holds for all $s \in \mathcal{S}, a \in \mathcal{A}$.*

Under this assumption, the modified standard Q-learning is given in Algorithm 1. Compared to the original version, the step-size α_k does not depend on the state-action pair in this version. With a suitable choice on the step-size, Algorithm 1 converges to the optimal Q^* with probability one.

Algorithm 1 Standard Q-Learning

```
1: Initialize Q_0 ∈ R^{|S||A|} randomly.
2: for iteration k = 0, 1, ... do
3:     Sample (s, a) ~ d_a(s)
4:     Sample s' ~ P_a(s, ·) and r_a(s, s')
5:     Update Q_{k+1}(s, a) = Q_k(s, a) + α_k{r_a(s, s') + γ max_{a∈A} Q_k(s', a) − Q_k(s, a)}
6: end for
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Theorem 1. *Assume that the step-sizes satisfy*

$$\alpha_k > 0, \quad \sum_{k=0}^{\infty} \alpha_k = \infty, \quad \sum_{k=0}^{\infty} \alpha_k^2 < \infty. \tag{5}$$

Then, $Q_k \to Q^*$ with probability one.

4 Analysis of Q-learning from Switching System Theory

In this section, we study a switching system-based ODE model of Q-learning and prove the convergence of Q-learning in Theorem 1 based on the switching system analysis.

We first introduce the following compact notations:

$$P := \begin{bmatrix} P_1 \\ \vdots \\ P_{|\mathcal{A}|} \end{bmatrix} \in \mathbb{R}^{|\mathcal{S}|\times|\mathcal{S}||\mathcal{A}|}, \quad R := \begin{bmatrix} R_1 \\ \vdots \\ R_{|\mathcal{A}|} \end{bmatrix} \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}, \quad Q := \begin{bmatrix} Q_1 \\ \vdots \\ Q_{|\mathcal{A}|} \end{bmatrix} \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|},$$

$$D_a := \begin{bmatrix} d_a(1) & & \\ & \ddots & \\ & & d_a(|\mathcal{S}|) \end{bmatrix} \in \mathbb{R}^{|\mathcal{S}|\times|\mathcal{S}|}, \quad D := \begin{bmatrix} D_1 & & \\ & \ddots & \\ & & D_{|\mathcal{A}|} \end{bmatrix} \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|\times|\mathcal{S}||\mathcal{A}|},$$

where $Q_a = Q(\cdot, a) \in \mathbb{R}^{|\mathcal{S}|}, a \in \mathcal{A}$ and $R_a(s) := \mathbb{E}[r_a(s, s')|s, a]$. Note that D is a nonsingular diagonal matrix with strictly positive diagonal elements. Using the notation introduced, the update in Algorithm 1 can be written as

$$Q_{k+1} = Q_k + \alpha_k\{(e_a \otimes e_s)(e_a \otimes e_s)^T R + \gamma(e_a \otimes e_s)(e_{s'})^T \max_{a\in\mathcal{A}} Q(\cdot, a) - (e_a \otimes e_s)(e_a \otimes e_s)^T Q\},$$

where $e_s \in \mathbb{R}^{|\mathcal{S}|}$ and $e_a \in \mathbb{R}^{|\mathcal{A}|}$ are s-th basis vector (all components are 0 except for the s-th component which is 1) and a-th basis vector, respectively. For any deterministic policy, $\pi : \mathcal{S} \to \mathcal{A}$, we define the corresponding distribution vector $\vec{\pi}(s) := e_{\pi(s)} \in \Delta_{|\mathcal{S}|}$, where $\Delta_{|\mathcal{S}|}$ is the set of all probability distributions over $\mathcal{S}$, and the matrix

$$\Pi_\pi := \begin{bmatrix} \vec{\pi}(1)^T \otimes e_1^T \\ \vec{\pi}(2)^T \otimes e_2^T \\ \vdots \\ \vec{\pi}(|\mathcal{S}|)^T \otimes e_{|\mathcal{S}|}^T \end{bmatrix} \in \mathbb{R}^{|\mathcal{S}|\times|\mathcal{S}||\mathcal{A}|}.$$

Denoting $\pi_Q(s) := \arg\max_{a\in\mathcal{A}} e_s^T Q_a \in \mathcal{A}$, the update can be further simplified as

$$Q_{k+1} = Q_k + \alpha_k\{DR + \gamma DP\Pi_{\pi_{Q_k}} Q_k - DQ_k + \varepsilon_{k+1}\}, \tag{6}$$

where $\varepsilon_{k+1} = (e_a \otimes e_s)(e_a \otimes e_s)^T R + \gamma(e_a \otimes e_s)(e_{s'})^T \Pi_{\pi_{Q_k}} Q_k - (e_a \otimes e_s)(e_a \otimes e_s)^T Q_k - (DR + \gamma DP\Pi_{\pi_{Q_k}} Q_k - DQ_k)$. We note that, for any $\pi \in \Theta$, $P\Pi_\pi$ is the state-action pair transition probability matrix under the deterministic policy π. Using the Bellman equation

$$(\gamma DP\Pi_{\pi_{Q^*}} - D)Q^* + DR = 0,$$

(6) can be rewritten as

$$(Q_{k+1} - Q^*) = (Q_k - Q^*) + \alpha_k\{(\gamma DP\Pi_{\pi_{Q_k}} - D)(Q_t - Q^*) + \gamma DP(\Pi_{\pi_{Q_k}} - \Pi_{\pi_{Q^*}})Q^* + \varepsilon_{k+1}\}. \tag{7}$$

As discussed in Section 2.4, the convergence of (7) can be analyzed by evaluating the stability of the corresponding continuous-time ODE

$$\frac{d}{dt}(Q_t - Q^*) = (\gamma DP\Pi_{\pi_{Q_t}} - D)(Q_t - Q^*) + \gamma DP(\Pi_{\pi_{Q_t}} - \Pi_{\pi_{Q^*}})Q^*,\ Q_0 - Q^* = z \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}, \tag{8}$$

which is a switching system. More precisely, if we define a one-to-one map $\psi : \Theta \to \{1, 2, \ldots, |\Theta|\}$, where Θ is the set of all deterministic policies, $x_t := Q_t - Q^*$, and

$$(A_{\psi(\pi)}, b_{\psi(\pi)}) := (\gamma DP\Pi_\pi - D, \gamma DP(\Pi_\pi - \Pi_{\pi_{Q^*}})Q^*)$$

for all $\pi \in \Theta$, then (8) can be represented by the affine switching system

$$\frac{d}{dt}x_t = A_{\sigma(x_t)}x_t + b_{\sigma(x_t)}, \quad x_0 = z \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}, \tag{9}$$

where, $\sigma : \mathbb{R}^{|\mathcal{S}||\mathcal{A}|} \to \{1, 2, \ldots, |\Theta|\}$ is a state-feedback switching policy defined by $\sigma(x_t) := \psi(\pi_{Q_t})$, $\pi_{Q_t}(s) = \arg\max_{a\in\mathcal{A}} e_s^T Q_{t,a}$.

Moreover, we establish the existence and uniqueness of its solution, which follows from the global Lipschitz continuity of the affine mapping. The proof is given in Appendix.

Proposition 1. *Define $f(\theta) = (\gamma DP\Pi_{\pi_\theta} - D)\theta$. Then, f is globally Lipschitz continuous.*

Invoking Lemma 1, we then have the following result

Proposition 2. *The switching system (9) has a unique solution for all $t \geq 0$ and $x(0) \in \mathbb{R}^n$.*

Note that proving the global asymptotic stability of (9) without the affine term is relevantly straightforward based on existing results, e.g., [10, Theorem 8]. However, with the affine term, the proof is no longer trivial with the existing approaches in switching system theories. In what follows, we show that by exploiting the special structure of the switching system and policy associated with the Q-learning update rule, the global asymptotic stability can still be proved.

We first establish the asymptotic stability of the corresponding linear switching system.

Lemma 5. *Consider the affine switching system* (9)*. The origin of the linear switching system*

$$\frac{d}{dt}x_t = A_{\sigma_t}x_t,$$

is the unique globally asymptotically stable equilibrium point under arbitrary switchings, σ_t.

The proof follows by applying Lemma 3 with $L = I, \bar{A}_\sigma = A_\sigma$. We defer the proof to the Appendix.

We are now in position to prove the asymptotic stability of (9) associated with Q-learning.

Theorem 2. *The origin is the unique globally asymptotically stable equilibrium point of the affine switching system* (9)*.*

Proof. The basic idea of the proof is to find systems whose trajectories lower and upper bound the trajectory of (9) by the vector comparison principle Lemma 2. Then, by proving asymptotic stability of the two comparison systems, we can prove the asymptotic stability of (9). Here, we only provide a brief sketch of the proof, and the full proof is deferred to Appendix. Since each element of $\Pi_{\pi_{Q^*}}Q^*$ takes the maximum value across a, it is clear that $(\Pi_{\pi_{Q_t}} - \Pi_{\pi_{Q^*}})Q^* \leq 0$ holds, where the inequality is element-wise. Moreover, since γDP has nonnegative elements, $\gamma DP(\Pi_{\pi_{Q_t}} - \Pi_{\pi_{Q^*}})Q^* \leq 0$ holds. Therefore, we have $(\gamma D_\beta P\Pi_{\pi_{Q_t}} - D)(Q_t - Q^*) + \gamma DP(\Pi_{\pi_{Q_t}} - \Pi_{\pi_{Q^*}})Q^* \leq (\gamma DP\Pi_{\pi_{Q_t}} - D)(Q_t - Q^*)$ for all $t \in \mathbb{R}_+$. Now, consider the switching system, which we refer to as an upper comparison system: $\frac{d}{dt}(Q_t^u - Q^*) = (\gamma DP\Pi_{\pi_{Q_t^u}} - D)(Q_t^u - Q^*), \quad Q_0^u - Q^* > z \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$. We can prove that the vector function f associated with the above system is quasi-monotone increasing. By the comparison principle, Lemma 2, $Q_t - Q^* \leq Q_t^u - Q^*$ holds for every $t \in \mathbb{R}_+$, where $Q_t^u - Q^*$ is the solution of the upper comparison system. By Lemma 5, the origin of the above switching system is globally asymptotically stable even under arbitrary switchings. Therefore, $Q_t - Q^*$ is asymptotically upper bounded by the zero vector as $t \to \infty$. On the other hand, we have

$$\begin{aligned}(\gamma DP\Pi_{\pi_{Q_t}} - D)(Q_t - Q^*) + \gamma DP(\Pi_{\pi_{Q_t}} - \Pi_{\pi_{Q^*}})Q^* =&(\gamma DP\Pi_{\pi_{Q_t}} - D)Q_t + DR \\ \geq&(\gamma DP\Pi_{\pi_{Q^*}} - D)Q_t + DR \\ =&(\gamma DP\Pi_{\pi_{Q^*}} - D)(Q_t - Q^*),\end{aligned}$$

where the first inequality is due to $\gamma DP\Pi_{\pi_{Q_t}}Q_t \geq \gamma DP\Pi_{\pi_{Q^*}}Q_t$, and the second equality uses $DQ^* = \gamma DP\Pi_{\pi_{Q^*}}Q^* + DR$. Therefore, consider the following linear system called the lower comparison system: $\frac{d}{dt}(Q_t^l - Q^*) = (\gamma DP\Pi_{Q^*} - D)(Q_t^l - Q^*), \quad Q_0^l - Q^* < z \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$. The origin of the above linear system is globally asymptotically stable equilibrium point by Lemma 5. Moreover, we can prove that the vector function f associated with the original switching system (9) is quasi-monotone increasing. Again, we invoke the vector comparison principle, Lemma 2, to prove the inequality $Q_t^l - Q^* \leq Q_t - Q^*$ for all $t \geq 0$, where $Q_t^l - Q^*$ is the solution of the lower comparison system. Therefore, $Q_t - Q^*$ is asymptotically lower bounded by the zero vector as $t \to \infty$. Combining the bounds, we conclude that $Q_t - Q^* \to 0$ as $t \to \infty$. This completes the proof of Theorem 2. □

Based on the results, we can now apply the Borkar and Meyn theorem, Lemma 4, to prove Theorem 1. The proof follows typical routines of the ODE approaches [1], thus omitted here due to the space limit and deferred to the Appendix.

5 Conclusion

In this paper, we studied the standard Q-learning algorithm through the switching system perspective, and provided a simple proof for the asymptotic convergence of Q-learning by leveraging existing theory on the stability of linear switching systems and comparison principles. The switching system approach also provides a convenient tool for analysis of other Q-learning variants, and shed light on the underlying dynamics of RL algorithms. For future work, we would like to investigate the non-asymptotic convergence of Q-learning algorithms based on discrete-time stochastic switching system models.

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Appendix

A Technical Proofs

A.1 Proof of Lemma 2

For convenience, we restate the lemma again.
Vector comparison principle: Suppose that $\overline{f}$ and $\underline{f}$ are globally Lipschitz continuous so that the corresponding O.D.Es admit unique solutions in the conventional sense. Let v_t be a solution of the system

$$\frac{d}{dt}x_t = \overline{f}(x_t), \quad x_0 \in \mathbb{R}^n, \forall \quad t \geq 0,$$

assume that $\overline{f}$ is quasi-monotone increasing, and let v_t be a solution of the system

$$\frac{d}{dt}v_t = \underline{f}(v_t), \quad v_0 < x_0, \quad \forall t \geq 0, \tag{10}$$

where $\underline{f}(v) \leq \overline{f}(v)$ holds for any $v \in \mathbb{R}^n$. Then, $v_t \leq x_t$ for all $t \geq 0$.

We simplify and summarize the ideas of the proofs in the literature, [14, page 112],[4, Theorem 3.2.], in the following proof.

Proof: Instead of (10), first consider

$$\frac{d}{dt}v_\varepsilon(t) = \underline{f}(v_\varepsilon(t)) - \varepsilon \mathbf{1}_n, \quad v_\varepsilon(0) < x(0), \quad \forall t \geq 0$$

where $\varepsilon > 0$ is a sufficiently small real number and $\mathbf{1}_n$ is a vector where all elements are ones, where we use a different notation for the time index for convenience. Suppose that the statement is not true, and let

$$t^* := \inf\{t \geq 0 : \exists i \text{ such that } v_{\varepsilon,i}(t) > x_i(t)\} < \infty,$$

and let i be such index. By the definition of t^*, we have that $v_{\varepsilon,i}(t^*) = x_i(t^*)$ and $v_{\varepsilon,j}(t^*) \leq x_j(t^*)$ for any $j \neq i$. Then, since $\overline{f}$ is quasi-monotone increasing, we have

$$\overline{f}_i(v_\varepsilon(t^*)) \leq \overline{f}_i(x(t^*)). \tag{11}$$

On the other hand, by the definition of t^*, there exists a small $\delta > 0$ such that

$$v_{\varepsilon,i}(t^* + \Delta t) > x_i(t^* + \Delta t)$$

for all $0 < \Delta t < \delta$. Dividing both sides by Δt and taking the limit $\Delta t \to 0$, we have

$$\dot{v}_{\varepsilon,i}(t^*) \geq \dot{x}_i(t^*) = \overline{f}_i(x(t^*)). \tag{12}$$

By the hypothesis, it holds that

$$\frac{d}{dt}v_\varepsilon(t) = \underline{f}(v_\varepsilon(t)) - \varepsilon\mathbf{1}_n < \underline{f}(v_\varepsilon(t)) \leq \overline{f}(v_\varepsilon(t))$$

holds for all $t \geq 0$. The inequality implies $\dot{v}_{\varepsilon,i}(t) < \overline{f}_i(v_\varepsilon(t))$, which in combination with (12) leads to $\overline{f}_i(v_\varepsilon(t^*)) > \overline{f}_i(x(t^*))$. However, it contradicts with (11). Therefore, $v_\varepsilon(t) \leq x(t)$ holds for all $t \geq 0$. Since the solution $v_\varepsilon(t)$ continuously depends on $\varepsilon > 0$ [14, Chap. 13], taking the limit $\varepsilon \to 0$, we conclude $v_0(t) \leq x(t)$ holds for all $t \geq 0$. This completes the proof.

A.2 Proof of Proposition 1

The proof is completed by the inequalities

$$\begin{aligned}\|f(x) - f(y)\|_\infty =& \|(\gamma DP\Pi_{\pi_x} - D)x - (\gamma DP\Pi_{\pi_y} - D)y\|_\infty \\ \leq& \|\gamma DP\|_\infty \|\Pi_{\pi_x}x - \Pi_{\pi_y}y\|_\infty + \|D\|_\infty\|x - y\|_\infty\end{aligned}$$

$$
\begin{aligned}
&= \|\gamma DP\|_\infty \max_{s\in\mathcal{S}} |\max_{a\in\mathcal{A}} x_a(s) - \max_{a\in\mathcal{A}} y_a(s)| + \|D\|_\infty \|x-y\|_\infty \\
&\le \|\gamma DP\|_\infty \max_{s\in\mathcal{S}} \max_{a\in\mathcal{A}} |x_a(s) - y_a(s)| + \|D\|_\infty \|x-y\|_\infty \\
&= \|\gamma DP\|_\infty \|x-y\|_\infty + \|D\|_\infty \|x-y\|_\infty \\
&\le (\|\gamma DP\|_\infty + \|D\|_\infty) \|x-y\|_\infty,
\end{aligned}
$$

indicating that f is globally Lipschitz continuous with respect to the $\|\cdot\|_\infty$ norm. This completes the proof.

A.3 Proof of Proposition 2

Note that using $(\gamma DP\Pi_{\pi_{Q^*}} - D)Q^* + DR = 0$, the solution of (8) is equivalent to the solution of the following system up to a constant shifting:

$$
\frac{d}{dt}Q_t = (\gamma DP\Pi_{\pi_{Q_t}} - D)Q_t + DR, \; Q_0 = z \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|},
$$

which can be expressed as $\frac{d}{dt}Q_t = f(Q_t)$ with $f(\theta) = (\gamma DP\Pi_{\pi_\theta} - D)\theta$. By Proposition 1, its solution exists and unique. This completes the proof.

A.4 Proof of Lemma 5

We apply Lemma 3 with $L = I, \bar{A}_\sigma = A_\sigma$. In this case, the condition, $LA_\sigma = \bar{A}_\sigma L$ holds. It remains to prove the strictly negative row dominating diagonal property. For notational convenience, we definte $\Pi_\sigma, \sigma \in \mathcal{M}$ as $\Pi_{\pi_{Q_t^B}}$ such that $\sigma = \psi(\pi_{Q_t^B})$. Letting $n = |\mathcal{S}||\mathcal{A}|$, the property is proved by

$$
\begin{aligned}
&[A_\sigma]_{ii} + \sum_{j\in\{1,2,\ldots,n\}\setminus\{i\}} |[A_\sigma]_{ij}| \\
=& [D]_{ii}[\gamma P\Pi_\sigma - I]_{ii} \\
&+ \sum_{j\in\{1,2,\ldots,n\}\setminus\{i\}} [D]_{ii}|[\gamma P\Pi_\sigma - I]_{ij}| \\
\le& [\gamma P\Pi_\sigma - I]_{ii} + \sum_{j\in\{1,2,\ldots,n\}\setminus\{i\}} |[\gamma P\Pi_\sigma - I]_{ij}| \\
=& [\gamma P\Pi_\sigma]_{ii} - 1 + \sum_{j\in\{1,2,\ldots,n\}\setminus\{i\}} |[\gamma P\Pi_\sigma]_{ij}| \\
=& [\gamma P\Pi_\sigma]_{ii} + \sum_{j\in\{1,2,\ldots,n\}\setminus\{i\}} |[\gamma P\Pi_\sigma]_{ij}| - 1 \\
=& \gamma - 1 \\
<& 0, \quad \forall \sigma \in \mathcal{M},
\end{aligned}
$$

which proves the global asymptotic stability. □

B Proof of Theorem 1

First of all, note that the affine switching system model in (9) corresponds to the ODE model, $\frac{d}{dt}x_t = f(x_t)$, that appears in Assumption 1. The proof is completed by examining all the statements in Assumption 1. In the following, we itemize the proofs of the statements in Assumption 1 in the same order.

1. Q-learning in (7) can be expressed as the stochastic recursion in (4) with

$$
f(\theta) = (\gamma DP\Pi_{\pi_\theta} - D)\theta + \gamma DP(\Pi_{\pi_\theta} - \Pi_{\pi_{Q^*}})Q^*.
$$

To prove the first statement of Assumption 1, we note that

$$
\frac{f(c\theta)}{c} = \frac{(\gamma DP\Pi_{\pi_{c\theta}} - D)c\theta + \gamma DP(\Pi_{\pi_{c\theta}} - \Pi_{\pi_{Q^*}})Q^*}{c}
$$

$$=(\gamma DP\Pi_{\pi_\theta} - D)\theta + \frac{\gamma DP(\Pi_{\pi_\theta} - \Pi_{\pi_{Q^*}})Q^*}{c},$$

where the last equality is due to the homogeneity of the policy, $\pi_{c\theta}(s) = \arg\max_{a\in\mathcal{A}} e_s^T c\theta_a = \arg\max_{a\in\mathcal{A}} e_s^T \theta_a$. By taking the limit, we have

$$\begin{aligned}\lim_{c\to\infty} \frac{f(c\theta)}{c} =& (\gamma DP\Pi_{\pi_\theta} - D)\theta \\ &+ \lim_{c\to\infty} \frac{\gamma DP(\Pi_{\pi_\theta} - \Pi_{\pi_{Q^*}})Q^*}{c} \\ =& (\gamma DP\Pi_{\pi_\theta} - D)\theta = f_\infty(\theta).\end{aligned}$$

Moreover, f is globally Lipschitz continuous according to the inequalities

$$\begin{aligned}&\|f(x) - f(y)\|_\infty \\ =&\|(\gamma DP\Pi_{\pi_x} - D)x - (\gamma DP\Pi_{\pi_y} - D)y\|_\infty \\ \le&\|\gamma DP\|_\infty\|\Pi_{\pi_x}x - \Pi_{\pi_y}y\|_\infty + \|D\|_\infty\|x-y\|_\infty \\ =&\|\gamma DP\|_\infty \max_{s\in\mathcal{S}} |\max_{a\in\mathcal{A}} x_a(s) - \max_{a\in\mathcal{A}} y_a(s)| \\ &+ \|D\|_\infty\|x-y\|_\infty \\ \le&\|\gamma DP\|_\infty \max_{s\in\mathcal{S}} \max_{a\in\mathcal{A}} |x_a(s) - y_a(s)| \\ &+ \|D\|_\infty\|x-y\|_\infty \\ =&\|\gamma DP\|_\infty\|\Pi_{\pi_{|x-y|}}(|x-y|)\|_\infty + \|D\|_\infty\|x-y\|_\infty \\ \le&\|\gamma DP\|_\infty\|\Pi_{\pi_{|x-y|}}\|_\infty\|x-y\|_\infty + \|D\|_\infty\|x-y\|_\infty \\ \le&\left(\|\gamma DP\|_\infty \max_{\pi\in\Theta}\|\Pi_\pi\|_\infty + \|D\|_\infty\right)\|x-y\|_\infty,\end{aligned}$$

implying that f is globally Lipschitz continuous with the parameter $\|\gamma DP\|_\infty \max_{\pi\in\Theta}\|\Pi_\pi\|_\infty + \|D\|_\infty$. Therefore, the proof is completed.

2. The second statement of Assumption 1 is directly proved by **Claim** in the proof of Theorem 2.
3. The third statement of Assumption 1 is directly proved by Theorem 2.
4. Next, we prove the remaining parts. If we define $M_k := \sum_{i=0}^k \varepsilon_i$, then M_k is Martingale as

$$\begin{aligned}&\mathbb{E}[M_{k+1}|\mathcal{G}_k] \\ =&\mathbb{E}\left[\sum_{i=0}^{k+1}\varepsilon_i\middle|(\varepsilon_i,\theta_i)_{i=1}^k\right] \\ =&\mathbb{E}[\varepsilon_{k+1}|(\varepsilon_i,\theta_i)_{i=1}^k] + \mathbb{E}\left[\sum_{i=0}^{k}\varepsilon_i\middle|(\varepsilon_i,\theta_i)_{i=1}^k\right] \\ =&\mathbb{E}\left[\sum_{i=0}^{k}\varepsilon_i\middle|(\varepsilon_i,\theta_i)_{i=1}^k\right] = \sum_{i=0}^k \varepsilon_i = M_k\end{aligned}$$

and ε_k is a Martingale difference sequence. Moreover, it can be easily proved that the fourth condition of Assumption 1 is satisfied. Therefore, the fourth condition is met. □

C Proof of Theorem 2

The basic idea of the proof is to find systems whose trajectories lower and upper bounds the trajectory of (9) by the vector comparison principle. Then, by proving the asymptotic stability of the two comparison systems, we can prove the asymptotic stability of (9).

Since each element of $\Pi_{\pi_{Q^*}}Q^*$ takes the maximum value across a, it is clear that $(\Pi_{\pi_{Q_t}} - \Pi_{\pi_{Q^*}})Q^* \le 0$ holds, where the inequality is element-wise. Moreover, since γDP has nonnegative elements,

$\gamma DP(\Pi_{\pi_{Q_t}} - \Pi_{\pi_{Q^*}})Q^* \le 0$ holds. Therefore, we have $(\gamma D_\beta P\Pi_{\pi_{Q_t}} - D)(Q_t - Q^*) + \gamma DP(\Pi_{\pi_{Q_t}} - \Pi_{\pi_{Q^*}})Q^* \le (\gamma DP\Pi_{\pi_{Q_t}} - D)(Q_t - Q^*) \le (\gamma DP\Pi_{\pi_{Q_t - Q^*}} - D)(Q_t - Q^*)$ for all $t \in \mathbb{R}_+$. To proceed, define the vector functions

$$\begin{aligned}\overline{f}(y) =&(\gamma DP\Pi_{\pi_y} - D)y,\\ \underline{f}(z) =&(\gamma D_\beta P\Pi_{\pi_{z+Q^*}} - D)z + \gamma DP(\Pi_{\pi_{z+Q^*}} - \Pi_{\pi_{Q^*}})Q^*,\end{aligned}$$

and consider the systems

$$\begin{aligned}\frac{d}{dt}y_t &= \overline{f}(y_t), \quad y_0 > Q_0 - Q^*,\\ \frac{d}{dt}z_t &= \underline{f}(z_t), \quad z_0 = Q_0 - Q^*,\end{aligned}$$

for all $t \ge 0$. To apply Lemma 2, we will prove that $\overline{f}$ is quasi-monotone increasing. For any $z \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$, consider a nonnegative vector $p \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$ such that its ith element is zero. Then, for any $i \in \mathcal{S}$, we have

$$\begin{aligned}e_i^T\overline{f}(z+p) =&e_i^T(\gamma DP\Pi_{z+p} - D)(z+p)\\ =&\gamma e_i^T DP\Pi_{z+p}(z+p) - e_i^T Dz - e_i^T Dp\\ =&\gamma e_i^T DP\Pi_{z+p}(z+p) - e_i^T Dz\\ =&\gamma e_i^T DP\begin{bmatrix}\max_a(z_a(1)+p_a(1))\\ \max_a(z_a(2)+p_a(2))\\ \vdots\\ \max_a(z_a(|\mathcal{S}|)+p_a(|\mathcal{S}|))\end{bmatrix} - e_i^T Dz\\ \ge&\gamma e_i^T DP\begin{bmatrix}\max_a z_a(1)\\ \max_a z_a(2)\\ \vdots\\ \max_a z_a(|S|)\end{bmatrix} - e_i^T Dz\\ =&e_i^T\overline{f}(z),\end{aligned}$$

which proves the quasi-monotone increasing property.Therefore, we can apply Lemma 2. In particular, by Lemma 2, $Q_t - Q^* \le Q_t^u - Q^*$ holds for every $t \in \mathbb{R}_+$, where $Q_t^u - Q^*$ is the solution of the switching system, which we refer to as an upper comparison system

$$\frac{d}{dt}(Q_t^u - Q^*) = (\gamma DP\Pi_{\pi_{Q_t^u}} - D)(Q_t^u - Q^*), \quad Q_0^u - Q^* > Q_0 - Q^* \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|},$$

By Lemma 5, the origin of the above switching system is globally asymptotically stable even under arbitrary switching policies. Therefore, $Q_t - Q^*$ is asymptotically upper bounded by the zero vector as $t \to \infty$.

On the other hand, we have

$$\begin{aligned}(\gamma DP\Pi_{\pi_{Q_t}} - D)(Q_t - Q^*) + \gamma DP(\Pi_{\pi_{Q_t}} - \Pi_{\pi_{Q^*}})Q^* =& (\gamma DP\Pi_{\pi_{Q_t}} - D)Q_t + DR\\ \ge&(\gamma DP\Pi_{\pi_{Q^*}} - D)Q_t + DR = (\gamma DP\Pi_{\pi_{Q^*}} - D)(Q_t - Q^*),\end{aligned}$$

where the first inequality is due to $\gamma DP\Pi_{\pi_{Q_t}}Q_t \ge \gamma DP\Pi_{\pi_{Q^*}}Q_t$, and the second equality uses $DQ^* = \gamma DP\Pi_{\pi_{Q^*}}Q^* + DR$. Again, define the vector functions for lower comparison parts

$$\begin{aligned}\overline{f}(y) =&(\gamma DP\Pi_{\pi_y} - D)y + DR,\\ \underline{f}(z) =&(\gamma DP\Pi_{\pi_{Q^*}} - D)z + DR\end{aligned} \tag{13}$$

and consider the systems

$$\begin{aligned}\frac{d}{dt}y_t &= \overline{f}(y_t), \quad y_0 = Q_0,\\ \frac{d}{dt}z_t &= \underline{f}(z_t), \quad z_0 < Q_0,\end{aligned}$$

for all $t \geq 0$. Similarly, we can prove that $\overline{f}$ is quasi-monotone increasing, and invoke Lemma 2, to prove the inequality $Q_t^l - Q^* \leq Q_t - Q^*$ for all $t \geq 0$, where $Q_t^l - Q^*$ is the solution of the following linear system called the lower comparison system:

$$\frac{d}{dt}(Q_t^l - Q^*) = (\gamma DP\Pi_{Q^*} - D)(Q_t^l - Q^*), \quad Q_0^l - Q^* < Q_0 - Q^* \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}.$$

Note that the solution of this system differs from the solution of (13) by a constant shifting. The origin of the above linear system is globally asymptotically stable equilibrium point by Lemma 5. Therefore, $Q_t - Q^*$ is asymptotically lower bounded by the zero vector as $t \to \infty$. Combining the bounds, we conclude that $Q_t - Q^* \to 0$ as $t \to \infty$. This completes the proof of Theorem 2.